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BOUNDEDNESS OF MULTIPLE MARCINKIEWICZ INTEGRAL OPERATORS WITH ROUGH KERNELS

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Abstract. This paper is concerned with giving some rather weak size conditions implying the L^p boundedness of the multiple Marcinkiewicz integrals for some fixed $1 < p < \infty$, which essentially improve and extend some known results.

1. Introduction

Let $\mathbb{R}^N(N = m \text{ or } n)$, $N \geq 2$, be the N-dimensional Euclidean space and S^{N-1} be the unit sphere in $\mathbb{R}^N$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. For nonzero points $x \in \mathbb{R}^N$, we denote $x' = x/|x|$. For $m \geq 2$, $n \geq 2$, let Ω be homogeneous of degree zero, integrable on $S^{m-1} \times S^{n-1}$ and satisfy

$$(1.1) \qquad \int_{S^{m-1}} \Omega(x_1', x_2') d\sigma(x_1') = \int_{S^{n-1}} \Omega(x_1', x_2') d\sigma(x_2') = 0.$$

Suppose that

$$P_{N_1}(u) = \sum_{l=1}^{N_1} a_l u^l \qquad \text{and} \qquad P_{N_2}(v) = \sum_{l=1}^{N_2} b_l v^l$$

be two real polynomials on $\mathbb{R}$ with $P_{N_1}(0) = P_{N_2}(0) = 0$.

The multiple Marcinkiewicz integral operator $\mu_{\Omega,P}$ along the "polynomial curve" (P_{N_1}, P_{N_2}) is defined by

$$\mu_{\Omega,P}(f)(x_1, x_2) = \left(\int_0^\infty \int_0^\infty |F_{s,t}(x_1, x_2)|^2 \frac{dsdt}{s^3t^3} \right)^{1/2},$$

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where

$$F_{s,t}(x_1,\,x_2) = \int\int_{|y_1|\le s,\,|y_2|\le t} \frac{\Omega(y_1',\,y_2')}{|y_1|^{m-1}|y_2|^{n-1}} \times f(x_1 - P_{N_1}(|y_1|)y_1',\, x_2 - P_{N_2}(|y_2|)y_2')dy_1dy_2$$

for all $f \in \mathcal{S}\,(\mathbb{R}^m \times \mathbb{R}^n)$.

When $P_{N_1}(u) = u$ and $P_{N_2}(v) = v$, we denote $\mu_{\Omega,P}$ by μ_Ω. Obviously, the operator μ_Ω is a natural analogy of the high-dimensional Marcinkiewicz integral introduced by Stein [17]. It is well-known that the Marcinkiewicz integral is an important special case of the Littlewood-Paley-Stein functions and that it plays a key role in harmonic analysis. Ones can consult [6, 7, 14, 15, 16, 17, 18, 23, 24], among numerous references, for its development and applications. In particular, for the multiple Marcinkiewicz integral operator μ_Ω, Ding [9] first showed that if $\Omega \in L\,(\log^+ L)^2\,(S^{m-1} \times S^{n-1})$, that is,

$$\int\int_{S^{m-1}\times S^{n-1}} |\Omega(y_1',\,y_2')|\left(\log^+|\Omega(y_1',\,y_2')|\right)^2 d\sigma(y_1')d\sigma(y_2') < \infty,$$

then μ_Ω is bounded on $L^2(\mathbb{R}^m \times \mathbb{R}^n)$. In 2000, Chen, Ding, and Fan [2] proved that μ_Ω is bounded on L^p $(1 < p < \infty)$, provided that $\Omega \in L^q(S^{m-1} \times S^{n-1})(q > 1)$. Subsequently, Chen, Fan, and Ying [4] extended the result of [9] to any $p \in (1, \infty)$. In 2003, Hu, Lu, and Yan [13] proved that if for $\alpha > 1/2$, Ω satisfies the following condition

$$\sup_{\xi_1'\in S^{m-1},\,\xi_2'\in S^{n-1}} \int\int_{S^{m-1}\times S^{n-1}} |\Omega(y_1',\,y_2')| \times \left(\log\frac{1}{|\xi_1'\cdot y_1'|}\log\frac{1}{|\xi_2'\cdot y_2'|}\right)^\alpha d\sigma(y_1')d\sigma(y_2') < \infty, \tag{1.2}$$

then μ_Ω is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $1 + 1/(2\alpha) < p < 1 + 2\alpha$.

The condition (1.2) in the one-parameter case was originally defined in Walsh's paper [22] and developed by Grafakos and Stefanov [12] in the study of L^p-boundedness of Calderón-Zygmund singular integral operator. For the sake of simplicity, we denote that for $\alpha > 0$,

$$G_\alpha(S^{m-1} \times S^{n-1}) = \{\Omega \in L^1(S^{m-1} \times S^{n-1}) : \quad \Omega \quad \text{satisfies} \quad (1.2)\}.$$

Employing the ideas in [12], ones easily see that $L(\log^+ L)^2(S^{m-1} \times S^{n-1})$ and $G_\alpha(S^{m-1} \times S^{n-1})$ for $\alpha > 1$ do not contain each other, and $\bigcup_{q>1} L^q(S^{m-1} \times S^{n-1})$ is a proper subset of $G_\alpha(S^{m-1} \times S^{n-1})$ for any $\alpha > 0$, also, of $L(\log^+ L)^2(S^{m-1} \times S^{n-1})$.

The operator μ_Ω is closely related to the multiple singular integral operator T_Ω introduced by Fefferman and Stein [11], which naturally generalize Calderón-Zygmund [1] singular integral operator on one parameter, where

$$T_\Omega(f)(x_1, x_2) = \text{p.v.} \int\int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{\Omega(y_1', y_2')}{|y_1|^m |y_2|^n} f(x_1 - y_1, x_2 - y_2) dy_1 dy_2$$

with Ω satisfying the same conditions as in μ_Ω. In both T_Ω and μ_Ω, the singularity is along the diagonal $\{x_1 = y_1\}$ and $\{x_2 = y_2\}$. Recently, many problems in analysis have led one to consider singular integrals with singularity along more general sets. One of the principal motivations for the study of such operators is the requirements of several complex variables and large classes of "subelliptic" equations. We refer the readers to Stein's survey articles [19, 20] for more background information. In this paper, we focus our attentions on $\mu_{\Omega, P}$, which have singularity along sets of the form $\{x_1 = P_{N_1}(|y_1|)y_1'\}$ and $\{x_2 = P_{N_2}(|y_2|)y_2'\}$. In 2001, Chen, Ding, and Fan [3] proved that if $\Omega \in L^q(S^{m-1} \times S^{n-1})$ $(q > 1)$, then $\mu_{\Omega, P}$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$, $1 < p < \infty$, and the bound is independent of the coefficients of P_{N_1} and P_{N_2}. Later on, Ying [26] (resp., the author [25]) extended the result of [3] to the case $\Omega \in L(\log^+ L)^2(S^{m-1} \times S^{n-1})$ (resp., Ω belongs to certain block spaces).

A question that arises naturally is whether the general operator $\mu_{\Omega, P}$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ under condition (1.2) for $\alpha > 1/2$. Our next theorem will give a positive solution to this problem.

Theorem 1. *Let Ω be a homogeneous function of degree zero and satisfy* (1.1). *If $\Omega \in G_\alpha(S^{m-1} \times S^{n-1})$ for $\alpha > 1/2$, then $\mu_{\Omega, P}$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $1 + 1/(2\alpha) < p < 1 + 2\alpha$. And the bound is independent of the coefficients of the polynomials P_{N_1} and P_{N_2}.*

Remark 1. Theorem 1 is an essential improvement and extension over the results in [3] and [26]. And the result of [13] is a natural consequence of our result when $P_{N_1}(u) = u$ and $P_{N_2}(v) = v$.

In addition, the other two weaker conditions on Ω are that $\Omega \in L\log^+ L(S^{m-1} \times S^{n-1})$ and $\Omega \in G_{1/2}(S^{m-1} \times S^{n-1})$. By the ideas of [22], Chen, Fan and Ying [5] and Choi [8] obtained the $L^2(\mathbb{R}^m \times \mathbb{R}^n)$ boundedness of μ_Ω, provided that $\Omega \in L\log^+ L(S^{m-1} \times S^{n-1})$. And it is not difficult to verify that $L\log^+ L(S^{m-1} \times S^{n-1}) \subset G_{1/2}(S^{m-1} \times S^{n-1})$ (see Proposition 1 in Section 4). In our next theorem, it will be show that

$\Omega \in G_{1/2}(S^{m-1} \times S^{n-1})$ suffices to imply the $L^2(\mathbb{R}^m \times \mathbb{R}^n)$ boundedness of μ_Ω.

THEOREM 2. *Suppose that Ω is a homogeneous function of degree zero and satisfies* (1.1). *Then μ_Ω is bounded on $L^2(\mathbb{R}^m \times \mathbb{R}^n)$, provided that $\Omega \in G_{1/2}(S^{m-1} \times S^{n-1})$.*

REMARK 2. Since for $\alpha > 1/2$, $G_\alpha(S^{m-1} \times S^{n-1}) \subset G_{1/2}(S^{m-1} \times S^{n-1})$, which is a proper inclusion, and the method of [13] does not work for the case $\Omega \in G_{1/2}(S^{m-1} \times S^{n-1})$. Thus Theorem 2 essentially improve the corresponding result of [13] for $p = 2$. An interesting problem is whether $\Omega \in G_{1/2}(S^{m-1} \times S^{n-1})$ also suffices to imply the L^2-boundedness of $\mu_{\Omega,\,P}$, moreover, the L^p-boundedness of $\mu_{\Omega,\,P}$ for $p \neq 2$.

This paper is organized as follows. In Section 2 we shall introduce some notations and give some technical lemmas. The proof of Theorem 1 will be given in Section 3. Finally, we shall prove Theorem 2 in Section 4. We remark that our some ideas in the proofs of our main results are taken from [10, 3, 13, 22], but our methods and techniques are more delicate and complex than that of [10, 3, 13, 22].

Throughout this paper, we always use the letter C to denote positive constants that may vary at each occurrence but are independent of the essential variables.

2. Main lemmas

Let us begin by introducing some notations. For given polynomials P_{N_1} and P_{N_2}, we denote

$$P_{\lambda_1}(u) = \sum_{l=0}^{\lambda_1} a_l u^l, \quad \text{and} \quad P_{\lambda_2}(v) = \sum_{l=0}^{\lambda_2} b_l v^l,$$

where $\lambda_1 \in \{0,\, 1,\, \dots,\, N_1\}$ and $\lambda_2 \in \{0,\, 1,\, \dots,\, N_2\}$ with $a_0 = b_0 = 0$. For $j,\, k \in \mathbb{Z}$ and $s,\, t \in \mathbb{R}_+$, we denote

$$B^{s,\,t}_{j,\,k} = \left\{(x_1,\, x_2) \in \mathbb{R}^m \times \mathbb{R}^n : 2^j s < |x_1| \le 2^{j+1} s,\, 2^k t < |x_2| \le 2^{k+1} t\right\}.$$

Let Ω be as in Theorem 1. For $\lambda_1 \in \{0,\, 1, \dots,\, N_1\}$ and $\lambda_2 \in \{0,\, 1, \dots,\, N_2\}$, we define the functions $\sigma^{s,t}_{j,k;\,\lambda_1,\lambda_2}$ and $|\sigma^{s,t}_{j,k;\,\lambda_1,\lambda_2}|$ by letting their

Fourier transforms be

$$
\begin{aligned}
(2.1) \qquad \widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\,\xi_2) \;&= \tfrac{1}{2^{j+k}st}\int\int_{B^{s,t}_{j,k}} \frac{\Omega(y_1',\,y_2')}{|y_1|^{m-1}|y_2|^{n-1}} \\
&\qquad \times e^{-i[P_{\lambda_1}(|y_1|)\xi_1\cdot y_1'+P_{\lambda_2}(|y_2|)\xi_2\cdot y_2']}dy_1dy_2,
\end{aligned}
$$

and

$$
\begin{aligned}
(2.2) \qquad |\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\,\xi_2)| \;&= \tfrac{1}{2^{j+k}st}\int\int_{B^{s,t}_{j,k}} \frac{|\Omega(y_1'\,y_2')|}{|y_1|^{m-1}|y_2|^{n-1}} \\
&\qquad \times e^{-i[P_{\lambda_1}(|y_1|)\xi_1\cdot y_1'+P_{\lambda_2}(|y_2|)\xi_2\cdot y_2']}dy_1dy_2.
\end{aligned}
$$

Then

$$
(2.3) \qquad s^{-1}t^{-1}F_{s,t}(x_1,\,x_2) = \sum_{j=-\infty}^{-1}\sum_{k=-\infty}^{-1} 2^{j+k}\sigma^{s,t}_{j,k;\,N_1,N_2} * f(x_1,\,x_2),
$$

and by definitions and (1.1), it is easy to see that for $\lambda_1 \in \{0,\,1,\,\ldots,\,N_1\}$ and $\lambda_2 \in \{0,\,1,\ldots,\,N_2\}$,

$$
\widehat{\sigma}^{s,t}_{j,k;\,0,\lambda_2}(\xi_1,\,\xi_2) = \widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,0}(\xi_1,\,\xi_2) = 0.
$$

It is also easy to see that

$$
\left\|\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}\right\|_\infty \le C \qquad \text{and} \qquad \left\||\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}|\right\|_\infty \le C
$$

hold uniformly for $j,\,k,\,s,\,t,\,\lambda_1$ and λ_2.

For all positive integers λ_1 and λ_2, we define the maximal functions by

$$
\sigma^*_{\lambda_1,\,\lambda_2}(f)(x_1,\,x_2) = \sup_{j,\,k\in\mathbb{Z}}\sup_{s,\,t>0}\left||\sigma^{s,t}_{j,k;\,\lambda_1,\lambda_2}| * f(x_1,x_2)\right|.
$$

Lemma 1. *For each pair λ_1 and λ_2, $\sigma^*_{\lambda_1,\lambda_2}$ is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n)$, $1 < p \le \infty$, and the bound is independent of the coefficients of P_{λ_1} and P_{λ_2}.*

The proof of Lemma 1 is similar to that of Proposition 2.1 in [3], we omit the details.

Lemma 2. *Let $s,\,t > 0$, $j,\,k \in \mathbb{Z}$ and $\Omega \in G_\alpha(S^{m-1}\times S^{n-1})$ for $\alpha > 1/2$. Then for each pair λ_1 and λ_2, there exist $C > 0$ such that*

(i)

$$
\begin{aligned}
&|\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\xi_2)-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2}(\xi_1,\xi_2)\\
&-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2-1}(\xi_1,\xi_2)+\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2-1}(\xi_1,\xi_2)|\\
\le\, & C\min\Big\{1,\ |2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|,\ |2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|,\\
&|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1||2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|\Big\};
\end{aligned}
\tag{2.4}
$$

(ii) *if* $|2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|>2^{\alpha\lambda_2}$, *then*

$$
\begin{aligned}
&|\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\xi_2)-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2}(\xi_1,\xi_2)|\\
\le\ & C|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|\min\Big\{1,\ \big(\log|2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|\big)^{-\alpha}\Big\},
\end{aligned}
\tag{2.5}
$$

and

$$
|\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\xi_2)|\le C\min\Big\{1,\ \Big(\log|2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|\Big)^{-\alpha}\Big\};
\tag{2.6}
$$

(iii) *if* $|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|>2^{\alpha\lambda_1}$, *then*

$$
\begin{aligned}
&|\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\xi_2)-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2-1}(\xi_1,\xi_2)|\\
\le\ & C|2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|\min\Big\{1,\ \big(\log|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|\big)^{-\alpha}\Big\},
\end{aligned}
\tag{2.7}
$$

and

$$
|\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\xi_2)|\le C\min\Big\{1,\ \Big(\log|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|\Big)^{-\alpha}\Big\};
\tag{2.8}
$$

(vi) *if* $|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|>2^{\alpha\lambda_1}$ *and* $|2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|>2^{\alpha\lambda_2}$, *then*

$$
\begin{aligned}
&|\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\xi_2)|\\
&\le C\min\Big\{1,\ \Big(\log|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|\Big)^{-\alpha}\Big(\log|2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|\Big)^{-\alpha}\Big\}.
\end{aligned}
\tag{2.9}
$$

Here C *are independent of* $j,\,k\in\mathbb{Z}$, $s,\,t>0$, $(\xi_1,\,\xi_2)\in\mathbb{R}^m\times\mathbb{R}^n$ *and the coefficients of* P_{λ_1} *and* P_{λ_2}.

Proof. (2.4) follows from the following inequality

$$\begin{aligned}
&|\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\xi_2)-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2}(\xi_1,\xi_2)-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2-1}(\xi_1,\xi_2)\\
&+\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2-1}(\xi_1,\xi_2)|\\
\le{}& C\int\int_{S^{m-1}\times S^{n-1}}|\Omega(y_1',\,y_2')|\\
&\times\Big|\int_1^2\int_1^2 e^{-i\{P_{\lambda_1-1}(2^jsr_1)\xi_1\cdot y_1'+P_{\lambda_2-1}(2^ktr_2)\xi_2\cdot y_2'\}}\\
&\times\Big[e^{-ia_{\lambda_1}2^{j\lambda_1}s^{\lambda_1}r_1^{\lambda_1}\xi_1\cdot y_1'}-1\Big]\\
&\times\Big[e^{-ib_{\lambda_2}2^{k\lambda_2}t^{\lambda_2}r_2^{\lambda_2}\xi_2\cdot y_2'}-1\Big]\,dr_1dr_2\Big|d\sigma(y_1')d\sigma(y_2').
\end{aligned}$$

To prove (2.5), we write

$$\begin{aligned}
&\Big|\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\,\xi_2)-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2}(\xi_1,\,\xi_2)\Big|\\
={}&\Big|\int\int_{S^{m-1}\times S^{n-1}}\Omega(y_1',\,y_2')\Big[\int_1^2 e^{-iP_{\lambda_2}(2^ktr_2)\xi_2\cdot y_2'}dr_2\Big]\\
&\times\Big[\int_1^2 e^{-iP_{\lambda_1-1}(2^jsr_1)\xi_1\cdot y_1'}\Big(e^{-ia_{\lambda_1}2^{j\lambda_1}s^{\lambda_1}r_1^{\lambda_1}\xi_1\cdot y_1'}-1\Big)\,dr_1\Big]\,d\sigma(y_1')d\sigma(y_2')\Big|.
\end{aligned}$$

By van der Corput lemma, we have

$$\Big|\int_1^2 e^{-iP_{\lambda_2}(2^ktr_2)\xi_2\cdot y_2'}dr_2\Big|\le C\left(2^{k\lambda_2}t^{\lambda_2}|b_{\lambda_2}||\xi_2||\xi_2'\cdot y_2'|\right)^{-1/\lambda_2}.$$

This together with the trivial estimate

$$\Big|\int_1^2 e^{-iP_{\lambda_2}(2^ktr_2)\xi_2\cdot y_2'}dr_2\Big|\le 1 \tag{2.10}$$

implies

$$\Big|\int_1^2 e^{-iP_{\lambda_2}(2^ktr_2)\xi_2\cdot y_2'}dr_2\Big|\le C\min\left\{1,\quad\left(\frac{2^{\alpha\lambda_2}|\xi_2'\cdot y_2'|^{-1}}{|2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|}\right)^{1/\lambda_2}\right\}.$$

Since $t/\log^a t$ is increasing in $(2^a,\,+\infty)$ for any $a>0$, we can deduce that for $\alpha>1/2$, if $|2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|>2^{\alpha\lambda_2}$, then

$$\Big|\int_1^2 e^{-iP_{\lambda_2}(2^ktr_2)\xi_2\cdot y_2'}dr_2\Big|\le C\min\left\{1,\quad\frac{\log^\alpha(2^{\alpha\lambda_2}|\xi_2'\cdot y_2'|^{-1})}{\log^\alpha(|2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|)}\right\}. \tag{2.11}$$

On the other hand, it is easy to see that

(2.12)
$$\left|\int_1^2 e^{-iP_{\lambda_1-1}(2^j sr_1)\xi_1\cdot y_1'}\left[e^{-ia_{\lambda_1}2^{j\lambda_1}s^{\lambda_1}r_1^{\lambda_1}\xi_1\cdot y_1'}-1\right]dr_1\right|\le C|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|,$$

Combing (2.10)-(2.12) with (1.2), we obtain (2.5).

Similarly, we can conclude (2.7).

It remains to prove (2.6), (2.8) and (2.9). Since

$$\begin{aligned}\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\,\xi_2) &= \int\int_{S^{m-1}\times S^{n-1}}\Omega(y_1',\,y_2')\left[\int_1^2 e^{-iP_{\lambda_1}(2^j sr_1)\xi_1\cdot y_1'}dr_1\right]\\ &\quad\times\left[\int_1^2 e^{-iP_{\lambda_2}(2^k tr_2)\xi_2\cdot y_2'}dr_2\right]d\sigma(y_1')d\sigma(y_2').\end{aligned}$$

Invoking (2.11) and the similar estimate

$$\begin{aligned}\left|\int_1^2 e^{-iP_{\lambda_1}(2^j sr_1)\xi_1\cdot y_1'}dr_1\right| &\le C\min\left\{1,\ \frac{\log^\alpha(2^{\alpha\lambda_1}|\xi_1'\cdot y_1'|^{-1})}{\log^\alpha(|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|)}\right\},\\ &\quad\text{if}\ \ |2^{j\lambda_1}s^{\lambda-1}a_{\lambda_1}\xi_1|>2^{\alpha\lambda_1},\end{aligned}$$

by (1.2) we can get (2.6), (2.8) and (2.9). This completes the proof of Lemma 2. □

Now we take two radial Schwartz functions $\phi_1\in\mathcal{S}(\mathbb{R}^m)$ and $\phi_2\in\mathcal{S}(\mathbb{R}^n)$ such that $\phi_i(r)\equiv 1$ for $|r|\le 1$ and $\phi_i(r)=0$ for $|r|>2$ $(i=1,\,2)$. Let $\varphi_i(r)=\phi_i(r^2)$ $(i=1,\,2)$ and define the measures $\{\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2}\}$ by

$$\begin{aligned}&\widehat{\tau}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\,\xi_2)\\ &= \widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\,\xi_2)\prod_{l=\lambda_1+1}^{N_1}\varphi_1(2^{jl}s^l a_l\xi_1)\prod_{l'=\lambda_2+1}^{N_2}\varphi_2(2^{kl'}t^{l'}b_{l'}\xi_2)\\ &\quad-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2}(\xi_1,\,\xi_2)\prod_{l=\lambda_1}^{N_1}\varphi_1(2^{jl}s^l a_l\xi_1)\prod_{l'=\lambda_2+1}^{N_2}\varphi_2(2^{kl'}t^{l'}b_{l'}\xi_2)\\ &\quad-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2-1}(\xi_1,\,\xi_2)\prod_{l=\lambda_1+1}^{N_1}\varphi_1(2^{jl}s^l a_l\xi_1)\prod_{l'=\lambda_2}^{N_2}\varphi_2(2^{kl'}t^{l'}b_{l'}\xi_2)\\ &\quad+\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2-1}(\xi_1,\,\xi_2)\prod_{l=\lambda_1}^{N_1}\varphi_1(2^{jl}s^l a_l\xi_1)\prod_{l'=\lambda_2}^{N_2}\varphi_2(2^{kl'}t^{l'}b_{l'}\xi_2),\end{aligned}$$

for $j,\,k\in\mathbb{Z}$, $s,\,t>0$, and $\lambda_1=1,\,2,\ldots,N_1$, and $\lambda_2=1,\,2,\ldots,N_2$, where we use the convention $\prod_{j\in\emptyset}A_j=1$.

Because $\widehat{\sigma}^{s,t}_{j,k;\,0,\lambda_2}(\xi_1,\,\xi_2)=\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,0}(\xi_1,\,\xi_2)=0$, it is easy to see that

(2.13)
$$s^{-1}t^{-1}F_{s,t}(x_1,\,x_2)=\sum_{\lambda_1=1}^{N_1}\sum_{\lambda_2=1}^{N_2}\sum_{j=-\infty}^{-1}\sum_{k=-\infty}^{-1}2^{j+k}\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2}*f(x_1,\,x_2).$$

And by Lemma 2, we have the following estimates for $\{\widehat{\tau}^{s,t}_{j,k;\,\lambda_1,\lambda_2}\}$.

LEMMA 3. *For* $\lambda_1=1,\,2,\ldots,N_1$, *and* $\lambda_2=1,\,2,\ldots,N_2$, $s,\,t>0$, $\alpha>1/2$,

(i) $|\widehat{\tau}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\,\xi_2)|\le C|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1||2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|$;

(ii) *if* $|2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|>2^{\alpha\lambda_2}$, *then*
$$|\widehat{\tau}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\,\xi_2)|\le C|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|\log^{-\alpha}|2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|;$$

(iii) *if* $|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|>2^{\alpha\lambda_1}$, *then*
$$|\widehat{\tau}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\,\xi_2)|\le C|2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|\log^{-\alpha}|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|;$$

(vi) *if* $|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|>2^{\alpha\lambda_1}$ *and* $|2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|>2^{\alpha\lambda_2}$, *then*
$$|\widehat{\tau}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\,\xi_2)|\le C\left(\log|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|\right)^{-\alpha}\left(\log|2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|\right)^{-\alpha}.$$

Here C *are independent of the coefficients of* P_{λ_1} *and* P_{λ_2}.

Proof. Write
$$\Pi_1(\lambda_1)=\prod_{l=\lambda_1+1}^{N_1}\varphi_1(2^{jl}s^la_l\xi_1)\ \text{ and }\ \Pi_2(\lambda_2)=\prod_{l'=\lambda_2+1}^{N_2}\varphi_2(2^{kl'}t^{l'}b_{l'}\xi_2).$$

Then
$$\Pi_1(\lambda_1-1)=\Pi_1(\lambda_1)\varphi_1(2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1)\quad\text{and}$$
$$\Pi_2(\lambda_2-1)=\Pi_2(\lambda_2)\varphi_2(2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2).$$

By these notations, we can write

(2.14)
$$\begin{aligned}\widehat{\tau}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\,\xi_2)\ &=\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\,\xi_2)\Pi_1(\lambda_1)\Pi_2(\lambda_2)\\ &\quad-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2}(\xi_1,\,\xi_2)\Pi_1(\lambda_1-1)\Pi_2(\lambda_2)\\ &\quad-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2-1}(\xi_1,\,\xi_2)\Pi_1(\lambda_1)\Pi_2(\lambda_2-1)\\ &\quad+\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2-1}(\xi_1,\,\xi_2)\Pi_1(\lambda_1-1)\Pi_2(\lambda_2-1).\end{aligned}$$

Thus, it is easy to see that

$$\begin{aligned}
&|\widehat{\tau}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\,\xi_2)|\\
\le\ &|\Pi_1(\lambda_1)\Pi_2(\lambda_2)|\\
&\times\Big\{\Big|\Big(\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2}-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2-1}+\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2-1}\Big)(\xi_1,\,\xi_2)\Big|\\
&+\Big|(\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2}-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2-1})(\xi_1,\,\xi_2)\Big|\,|1-\varphi_1(2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1)|\\
&+\Big|(\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2-1}-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2-1})(\xi_1,\,\xi_2)\Big|\,|1-\varphi_2(2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2)|\\
&+\Big|\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2-1}(\xi_1,\,\xi_2)\Big|\,|1-\varphi_1(2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1)||1-\varphi_2(2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2)|\Big\}.
\end{aligned}$$

Notice that

$$|1-\varphi_1(2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1)|\le C|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|, \tag{2.15}$$

and

$$|1-\varphi_2(2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2)|\le C|2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|, \tag{2.16}$$

by Lemma 2, we get (i).

Secondly,

$$\begin{aligned}
&\Big|\widehat{\tau}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\,\xi_2)\Big|\\
\le\ &\Big|\Big(\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2}-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2}\Big)(\xi_1,\,\xi_2)\Big|\,|\Pi_1(\lambda_1)\Pi_2(\lambda_2)|\\
&+\Big|\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2}(\xi_1,\,\xi_2)\Pi_1(\lambda_1)\Pi_2(\lambda_2)\Big|\,\Big|1-\varphi_1(2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1)\Big|\\
&+\Big|\Big(\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1,\lambda_2-1}-\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2-1}\Big)(\xi_1,\,\xi_2)\Big|\,|\Pi_1(\lambda_1)\Pi_2(\lambda_2-1)|\\
&+\Big|\widehat{\sigma}^{s,t}_{j,k;\,\lambda_1-1,\lambda_2-1}(\xi_1,\,\xi_2)\Pi_1(\lambda_1)\Pi_2(\lambda_2-1)\Big|\,\Big|1-\varphi_1(2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1)\Big|.
\end{aligned}$$

Observe that

$$\Pi_2(\lambda_2-1)=0,\qquad \text{if}\quad |2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|>2^{\alpha\lambda_2}. \tag{2.17}$$

Then using Lemma 2's (2.5) and (2.6), we obtain (ii).

Similarly, note that

$$\Pi_1(\lambda_1-1)=0,\qquad \text{if}\quad |2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|>2^{\alpha\lambda_1}, \tag{2.18}$$

we can get (iii).

Finally, (vi) follows from (2.9), (2.17) and (2.18) with (2.14). This completes the proof of Lemma 3. □

Also, by Lemma 1 and the definition of $\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2}$, we have

$$\Big\|\sup_{j,k\in\mathbb{Z}}\sup_{s,t>0}\Big|\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2}*f\Big|\Big\|_p\le C\|f\|_p, \tag{2.19}$$

for $\lambda_1 \in \{1, 2 \ldots, N_1\}$, $\lambda_2 \in \{1, 2, \ldots, N_2\}$ and $p \in (1, \infty)$, and the bounds are independent of the coefficients of the polynomials.

Applying (2.19), by the similar arguments to those used in Lemma 1 of [10], we can obtain the following lemma.

LEMMA 4. *For arbitrary functions* $\{g_{j,k}\}$,

$$\left\|\left(\sum_{j,\,k\in\mathbb{Z}}\left|\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2} * g_{j,k}\right|^2\right)^{1/2}\right\|_{p_0} \leq C\left\|\left(\sum_{j,\,k\in\mathbb{Z}}|g_{j,k}|^2\right)^{1/2}\right\|_{p_0}$$

for $1 < p_0 < \infty$, $\lambda_1 \in \{1, 2, \ldots, N_1\}$ *and* $\lambda_2 \in \{1, 2, \ldots, N_2\}$, *where* C *is independent of the coefficients of the polynomials* P_{λ_1} *and* P_{λ_2}.

3. Proof of Theorem 1

By Minkowski's inequality, it follows from (2.13) that

$$\begin{aligned}
&\mu_{\Omega,\,P}(f)(x_1,\,x_2)\\
&= \left(\int_0^\infty\int_0^\infty\left|\sum_{\lambda_1=1}^{N_1}\sum_{\lambda_2=1}^{N_2}\sum_{j=-\infty}^{-1}\sum_{k=-\infty}^{-1} 2^{j+k}\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2} * f(x_1,\,x_2)\right|^2\frac{dsdt}{st}\right)^{1/2}\\
&\leq \sum_{\lambda_1=1}^{N_1}\sum_{\lambda_1=2}^{N_2}\sum_{j=-\infty}^{-1}\sum_{k=-\infty}^{-1} 2^{j+k}\left(\int_0^\infty\int_0^\infty\left|\tau^{s,t}_{j,k;\lambda_1,\lambda_2} * f(x_1,\,x_2)\right|^2\frac{dsdt}{st}\right)^{1/2}\\
&\leq \sum_{\lambda_1=1}^{N_1}\sum_{\lambda_1=2}^{N_2}\sum_{j=-\infty}^{-1}\sum_{k=-\infty}^{-1} 2^{j+k}\left(\int_0^\infty\int_0^\infty\left|\tau^{s,t}_{0,0;\lambda_1,\lambda_2} * f(x_1,\,x_2)\right|^2\frac{dsdt}{st}\right)^{1/2}\\
&= \sum_{\lambda_1=1}^{N_1}\sum_{\lambda_1=2}^{N_2}\left(\int_0^\infty\int_0^\infty\left|\tau^{s,t}_{0,0;\lambda_1,\lambda_2} * f(x_1,\,x_2)\right|^2\frac{dsdt}{st}\right)^{1/2}\\
&= \sum_{\lambda_1=1}^{N_1}\sum_{\lambda_1=2}^{N_2}\left(\int_1^2\int_1^2\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left|\tau^{s,t}_{j,k;\lambda_1,\lambda_2} * f(x_1,\,x_2)\right|^2\frac{dsdt}{st}\right)^{1/2}.
\end{aligned}$$

Thus, to prove Theorem 1, it suffices to consider the $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ boundedness of The operator

(3.1)

$$\widetilde{\mu}_{\lambda_1,\,\lambda_2}(f)(x_1,\,x_2) = \left(\int_1^2\int_1^2\sum_{j,\,k\in\mathbb{Z}}\left|\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2} * f(x_1,\,x_2)\right|^2 dsdt\right)^{1/2}$$

for $\lambda_1 \in \{1, 2, \ldots, N_1\}$ and $\lambda_2 \in \{1, 2, \ldots, N_2\}$.

For each $j,\, k \in \mathbb{Z}$ and each fixed pair λ_1 and λ_2, by the definition of $\tau^{s,t}_{j,k;\lambda_1,\lambda_2}$, it is easy to see that if either $a_{\lambda_1} = 0$ or $b_{\lambda_2} = 0$, then $\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2} = 0$. Thus without loss of generality, we may assume $a_{\lambda_1} b_{\lambda_2} \neq 0$.

Take two radial Schwartz functions $\psi_1 \in \mathcal{S}(\mathbb{R}^m)$ and $\psi_2 \in \mathcal{S}(\mathbb{R}^n)$ such that

(i) $0 \le \psi_i \le 1, \quad i = 1,\, 2;$

(ii) $\operatorname{supp}(\psi_1) \subseteq \{2^{-\lambda_1} \le |\xi_1| \le 2^{\lambda_1}\}$ and $\operatorname{supp}(\psi_2) \subseteq \{2^{-\lambda_2} \le |\xi_2| \le 2^{\lambda_2}\}$;

(iii) $\sum_{d\in\mathbb{Z}} \big(\psi_1(2^{d\lambda_1} a_{\lambda_1}\xi_1)\big)^2 \equiv 1$ for all $\xi_1 \in \mathbb{R}^m \backslash \{0\}$ and $\sum_{l\in\mathbb{Z}} \big(\psi_2(2^{l\lambda_2} b_{\lambda_2}\xi_2)\big)^2 \equiv 1$ for all $\xi_2 \in \mathbb{R}^n \backslash \{0\}$.

Let $\psi_{1,\,d}(\xi_1) = \psi_1(2^{d\lambda_1} a_{\lambda_1}\xi_1)$ and $\psi_{2,\,l}(\xi_2) = \psi_2(2^{l\lambda_2} b_{\lambda_2}\xi_2)$. Define the multiplier operators Ψ^1_d and Ψ^2_l by

$$\widehat{\Psi^1_d f}(\xi_1) = \psi_{1,\,d}(\xi_1)\widehat{f}(\xi_1) \qquad \text{and} \qquad \widehat{\Psi^2_l f}(\xi_2) = \psi_{2,\,l}(\xi_2)\widehat{f}(\xi_2),$$

and $\Psi^1_d \otimes \Psi^2_l$ by

$$\big(\widehat{(\Psi^1_d \otimes \Psi^2_l)f}\big)(\xi_1,\, \xi_2) = \psi_{1,\,d}(\xi_1)\psi_{2,\,l}(\xi_2)\widehat{f}(\xi_1,\, \xi_2).$$

Then by checking the Fourier transforms, it is easy to see that for any test function f,

$$f(x_1,\, x_2) = \sum_{d\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \big((\Psi^1_d \otimes \Psi^2_l)^2 f\big)(x_1,\, x_2).$$

We can write

$$\begin{aligned} &\widetilde{\mu}_{\lambda_1,\,\lambda_2}(f)(x_1,\, x_2) \\ (3.2) \qquad = &\left(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\int_1^2\int_1^2 \left|\sum_{d\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \big(\Psi^1_{j-d}\otimes\Psi^2_{k-l}\big)\right.\right. \\ &\left.\left. \times \Big(\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2} * \big((\Psi^1_{j-d}\otimes\Psi^2_{k-l})f\big)\Big)(x_1,\, x_2)\right|^2 ds dt\right)^{1/2}. \end{aligned}$$

To establish the L^p-boundedness of $\widetilde{\mu}_{\lambda_1,\,\lambda_2}$, we first consider the mapping $\mathcal{G}$ defined by

(3.3)

$$\mathcal{G}: \quad \Big\{g^{s,t}_{j,k;\,d,l}\Big\}_{j,k\in\mathbb{Z};\,d,l\in\mathbb{Z}} \longrightarrow \left\{\sum_{d,l\in\mathbb{Z}} \big(\Psi^1_{j-d}\otimes\Psi^2_{k-l}\big)\Big(g^{s,t}_{j,k;\,d,l}\Big)(x_1,\, x_2)\right\}_{j,k\in\mathbb{Z}}.$$

By the same arguments as those used in [13, pp.78–81], we easily know that $\mathcal{G}$ is bounded from $l^q(L^p(\mathbb{R}^m\times\mathbb{R}^n)(L^2([1,\,2]\times[1,\,2])(l^2)))$ to $L^p(\mathbb{R}^m\times\mathbb{R}^n)(L^2([1,\,2]\times[1,\,2])(l^2))$ for each fixed $1<p<2$ and $1<q<p$, that is

(3.4)
$$\begin{aligned}&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_1^2\int_1^2\left|\sum_{d,l\in\mathbb{Z}}\left(\Psi^1_{j-d}\otimes\Psi^2_{k-l}\right)g^{s,t}_{j,k;\,d,l}\right|^2dsdt\right)^{1/2}\right\|_p^q\\ \leq\quad & C\sum_{d,l\in\mathbb{Z}}\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_1^2\int_1^2\left|g^{s,t}_{j,k;\,d,l}\right|^2dsdt\right)^{1/2}\right\|_p^q,\qquad 1<p<2,\end{aligned}$$

and bounded from $l^q(L^2([1,\,2]\times[1,\,2])(L^p(\mathbb{R}^m\times\mathbb{R}^n)(l^2)))$ to $L^p(\mathbb{R}^m\times\mathbb{R}^n)(L^2([1,\,2]\times[1,\,2])(l^2))$ for each fixed $2<p<\infty$ and $1<q<p'=p/(p-1)$, i.e.,

(3.5)
$$\begin{aligned}&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_1^2\int_1^2\left|\sum_{d,l\in\mathbb{Z}}\left(\Psi^1_{j-d}\otimes\Psi^2_{k-l}\right)g^{s,t}_{j,k;\,d,l}\right|^2dsdt\right)^{1/2}\right\|_p^q\\ \leq\quad & C\sum_{d,l\in\mathbb{Z}}\left(\int_1^2\int_1^2\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|g^{s,t}_{j,k;\,d,l}\right|^2\right)^{1/2}\right\|_p^2dsdt\right)^{q/2},\quad 2<p<\infty.\end{aligned}$$

Next for each fixed pair λ_1 and λ_2, we establish the $L^p(\mathbb{R}^m\times\mathbb{R}^n)$-boundedness of $\widetilde{\mu}_{\lambda_1,\,\lambda_2}$. We consider the following two cases:

Case 1. $1+1/(2\alpha)<p<2$. By (3.4), we have that for any $1<q<p$,

$$\begin{aligned}&\|\widetilde{\mu}_{\lambda_1,\,\lambda_2}(f)\|_p^q\\ &\leq C\sum_{d,l\in\mathbb{Z}}\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_1^2\int_1^2\left|\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2}*\left((\Psi^1_{j-d}\otimes\Psi^2_{k-l})f\right)\right|^2dsdt\right)^{1/2}\right\|_p^q.\end{aligned}$$

For each fixed $d,\,l\in\mathbb{Z}$, set

$$\begin{aligned}&I_{d,l}f(x_1,\,x_2)\\ &=\left(\sum_{j,k\in\mathbb{Z}}\int_1^2\int_1^2\left|\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2}*\left((\Psi^1_{j-d}\otimes\Psi^2_{k-l})f\right)(x_1,\,x_2)\right|^2dsdt\right)^{1/2}.\end{aligned}$$

By (2.19) and the definition of $\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2}$, we easily see that for any functions $\{h_{j,k}\}_{j,k\in\mathbb{Z}}$,

$$\left\|\sup_{j,k\in\mathbb{Z}}\sup_{s,t\in[1,\,2]}\left|\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2}*h_{j,k}\right|\right\|_{p_0}\le C\left\|\sup_{j,k\in\mathbb{Z}}|h_{j,k}|\right\|_{p_0},\qquad 1<p_0<\infty$$

and

$$\left\|\int_1^2\int_1^2\sum_{j,k\in\mathbb{Z}}\left|\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2}*h_{j,k}\right|dsdt\right\|_1\le C\left\|\sum_{j,k\in\mathbb{Z}}|h_{j,k}|\right\|_1.$$

Hence, by interpolation we get that for $1<p<2$,

(3.6)

$$\left\|\left(\int_1^2\int_1^2\sum_{j,k\in\mathbb{Z}}\left|\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2}*h_{j,k}\right|^2dsdt\right)^{1/2}\right\|_p\le C\left\|\left(\sum_{j,k\in\mathbb{Z}}|h_{j,k}|^2\right)^{1/2}\right\|_p.$$

On the other hand, by Plancherel's theorem, we have

$$\begin{aligned}
&\|I_{d,l}f\|_2^2\\
&=\int_1^2\int_1^2\sum_{j,k\in\mathbb{Z}}\int\int_{S^{m-1}\times S^{n-1}}|\widehat{f}(\xi_1,\,\xi_2)|^2|\psi_{1,\,j-d}(\xi_1)|^2|\psi_{2,\,k-l}(\xi_2)|^2\\
&\quad\times\left|\widehat{\tau}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\,\xi_2)\right|^2d\xi_1d\xi_2dsdt\\
&\le C\int_1^2\int_1^2\sum_{j,k\in\mathbb{Z}}\int\int_{E^{\lambda_1,\lambda_2}_{j-d,\,k-l}}|\widehat{f}(\xi_1,\,\xi_2)|^2\left|\widehat{\tau}^{s,t}_{j,k;\,\lambda_1,\lambda_2}(\xi_1,\,\xi_2)\right|^2d\xi_1d\xi_2dsdt,
\end{aligned}$$

where $E^{\lambda_1,\lambda_2}_{j-d,\,k-l}=\{(\xi_1,\,\xi_2)\in\mathbb{R}^m\times\mathbb{R}^n:2^{(d-j-1)\lambda_1}\le|a_{\lambda_1}\xi_1|\le 2^{(d-j+1)\lambda_1}$, $2^{(l-k-1)\lambda_2}\le|b_{\lambda_2}\xi_2|\le 2^{(l-k+1)\lambda_2}\}$.

Then by Lemma 3's (vi), we get that for $d>\alpha+1$ and $l>\alpha+1$,

(3.7)
$$\begin{aligned}
\|I_{d,l}f\|_2^2&\le C\int_1^2\int_1^2\sum_{j,k\in\mathbb{Z}}\int\int_{E^{\lambda_1,\lambda_2}_{j-d,\,k-l}}\left(\log|2^{j\lambda_1}s^{\lambda_1}a_{\lambda_1}\xi_1|\right)^{-2\alpha}\\
&\qquad\times\left(\log|2^{k\lambda_2}t^{\lambda_2}b_{\lambda_2}\xi_2|\right)^{-2\alpha}d\xi_1d\xi_2dsdt\\
&\le C(dl)^{-2\alpha}\|f\|_2^2.
\end{aligned}$$

Using interpolation between (3.6) and (3.7), it is easy to see that if $1<p<2$, then there exists $\varepsilon\in(2/(1+2\alpha),\ 1)$ such that

(3.8)
$$\|I_{d,l}f\|_p\le C(dl)^{-\varepsilon\alpha}\|f\|_p,\qquad d,\,l>\alpha+1.$$

Similarly, by using Lemma 3's (i), we can get that for $1 < p < 2$, there exists a $\theta > 0$ such that

$$\|I_{d,l}f\|_p \le C2^{(d+l)\theta}\|f\|_p, \qquad d,\, l \le \alpha+1. \tag{3.9}$$

By using Lemma 3's (ii) and (iii), it is easy to deduce that for $1 < p < 2$,

$$\|I_{d,\,l}\|_p \le Cd^{-\varepsilon\alpha}2^{l\theta}\|f\|_p, \qquad d > \alpha+1, \quad l \le \alpha+1, \tag{3.10}$$

and

$$\|I_{d,\,l}\|_p \le C2^{d\theta}l^{-\varepsilon\alpha}\|f\|_p, \qquad d \le \alpha+1, \quad l > \alpha+1, \tag{3.11}$$

where ε and θ is the same as that in (3.8) and (3.9), respectively.

And for fixed $p \in (1+1/(2\alpha),\ 2)$, we can choose $1 < q < p$ such that $q\varepsilon\alpha > 1$. Therefore, it follows from (3.8)-(3.11) that for $1 + 1/(2\alpha) < p < 2$,

$$\begin{aligned}
&\sum_{d,\,l\in\mathbb{Z}} \|I_{d,\,l}f\|_p^q \\
\le\ & C\Bigg\{\sum_{d\le\alpha+1}\sum_{l\le\alpha+1} 2^{q(d+l)\theta} + \sum_{d\le\alpha+1}\sum_{l>\alpha+1} 2^{qd\theta}l^{-q\varepsilon\alpha} \\
&+ \sum_{d>\alpha+1}\sum_{l\le\alpha+1} d^{-q\varepsilon\alpha}2^{ql\theta} + \sum_{d>\alpha+1}\sum_{l>\alpha+1} (dl)^{-q\varepsilon\alpha}\Bigg\}\|f\|_p^q \\
\le\ & C\|f\|_p^q,
\end{aligned}$$

which implies

$$\|\widetilde{\mu}_{\lambda_1,\,\lambda_2}(f)\|_p \le C\|f\|_p, \qquad 1+1/(2\alpha) < p < 2.$$

Case 2. $2 < p < 1+2\alpha$. By (3.5), we have that, for $2 < p < \infty$ and $1 < q < p' = p/(p-1)$,

$$\begin{aligned}
&\|\widetilde{\mu}_{\lambda_1,\,\lambda_2}(f)\|_p^q \\
&\le C\sum_{d,\,l\in\mathbb{Z}}\left(\int_1^2\int_1^2\left\|\left(\sum_{j,\,k\in\mathbb{Z}}\left|\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2} * \left((\Psi^1_{j-d}\otimes\Psi^2_{k-l})f\right)\right|^2\right)^{1/2}\right\|_p^2 dsdt\right)^{q/2}.
\end{aligned}$$

For each fixed $d,\, l \in \mathbb{Z}$, let

$$J^{s,t}_{d,\,l}f(x_1,\,x_2) = \left(\sum_{j,k\in\mathbb{Z}}\left|\tau^{s,t}_{j,k;\,\lambda_1,\lambda_2} * \left((\Psi^1_{j-d}\otimes\Psi^2_{k-l})f\right)(x_1,\,x_2)\right|^2\right)^{1/2}.$$

Then

$$\|\widetilde{\mu}_{\lambda_1,\,\lambda_2}(f)\|_p^q \le C \sum_{d,\,l\in\mathbb{Z}} \left(\int_1^2 \int_1^2 \left\|J_{d,\,l}^{s,t} f\right\|_p^2 dsdt\right)^{q/2}. \tag{3.12}$$

Applying Lemma 4 and the Littlewood-Paley theory (see [21, Chapter 4]), we have

$$\begin{aligned} \|J_{d,\,l}^{s,t} f\|_{p_0} &\le C \left\| \left(\sum_{j,k\in\mathbb{Z}} \left| \tau_{j,k;\,\lambda_1,\lambda_2}^{s,t} * \left((\Psi_{j-d}^1 \otimes \Psi_{k-l}^2) f \right) \right|^2 \right)^{1/2} \right\|_{p_0} \\ &\le C \left\| \left(\sum_{j,k\in\mathbb{Z}} \left| (\Psi_{j-d}^1 \otimes \Psi_{k-l}^2) f \right|^2 \right)^{1/2} \right\|_{p_0} \\ &\le C\|f\|_{p_0}, \qquad\qquad 1 < p_0 < \infty. \end{aligned} \tag{3.13}$$

Also, by Plancherel's theorem and Lemma 3, we can get that, for $s,\, t \in [1,\, 2]$,

$$\|J_{d,\,l}^{s,t} f\|_2 \le C 2^{d+l} \|f\|_2, \qquad \text{if} \;\; d \le \alpha+1, \;\; l \le \alpha+1; \tag{3.14}$$

$$\|J_{d,\,l}^{s,t} f\|_2 \le C d^{-\alpha} 2^{l} \|f\|_2, \qquad \text{if} \;\; d > \alpha+1, \;\; l \le \alpha+1; \tag{3.15}$$

$$\|J_{d,\,l}^{s,t} f\|_2 \le C 2^{d} l^{-\alpha} \|f\|_2, \qquad \text{if} \;\; d \le \alpha+1, \;\; l > \alpha+1; \tag{3.16}$$

$$\|J_{d,\,l}^{s,t} f\|_2 \le C (dl)^{-\alpha} \|f\|_2, \qquad \text{if} \;\; d > \alpha+1, \;\; l > \alpha+1. \tag{3.17}$$

And the constants C are independent of $s,\, t \in [1,\, 2]$.

Using interpolation theorem, the inequalities (3.13)-(3.17) show that, for any $2 < p < \infty$ and $2/(1+2\alpha) < \nu < 1$,

$$\|J_{d,\,l}^{s,t} f\|_p \le C 2^{\nu(d+l)} \|f\|_p, \qquad \text{if} \;\; d \le \alpha+1, \;\; l \le \alpha+1; \tag{3.18}$$

$$\|J_{d,\,l}^{s,t} f\|_p \le C 2^{\nu d} l^{-\nu\alpha} \|f\|_p, \qquad \text{if} \;\; d \le \alpha+1, \;\; l > \alpha+1; \tag{3.19}$$

$$\|J_{d,\,l}^{s,t} f\|_p \le C d^{-\nu\alpha} 2^{\nu l} \|f\|_p, \qquad \text{if} \;\; d > \alpha+1, \;\; l \le \alpha+1; \tag{3.20}$$

$$\|J_{d,\,l}^{s,t} f\|_p \le C (dl)^{-\nu\alpha} \|f\|_p, \qquad \text{if} \;\; d > \alpha+1, \;\; l > \alpha+1. \tag{3.21}$$

For each fixed $p \in (2,\ 1+2\alpha)$, we can choose $q \in (1,\ p')$ and $\nu \in (2/(1+2\alpha),\ 1)$ such that $q\nu\alpha > 1$. Then the inequalities (3.18)-(3.21) with (3.12) imply

$$\|\widetilde{\mu}_{\lambda_1,\,\lambda_2}(f)\|_p \le C\|f\|_p, \qquad\qquad 2 < p < 1+2\alpha.$$

This completes the proof of Theorem 1. □

4. A proposition and the proof of Theorem 2

Let us begin by proving the following proposition in this section.

Proposition 1. $L\log^+ L(S^{m-1}\times S^{n-1}) \subset G_{1/2}(S^{m-1}\times S^{n-1})$.

Proof. Let $\Omega \in L\log^+ L(S^{m-1}\times S^{n-1})$. Then

$$\int\!\!\int_{S^{m-1}\times S^{n-1}} |\Omega(x_1',\, x_2')|\log^+|\Omega(x_1',\, x_2')|d\sigma(x_1')d\sigma(x_2') < \infty.$$

To prove the proposition, it suffices to show that

$$\begin{aligned}&\int\!\!\int_{S^{m-1}\times S^{n-1}} |\Omega(x_1',\, x_2')| \\ &\times\left(\log\frac{1}{|\xi_1'\cdot x_1'|}\log\frac{1}{|\xi_2'\cdot x_2'|}\right)^{1/2} d\sigma(x_1')d\sigma(x_2') < \infty\end{aligned}$$

holds uniformly for $(\xi_1',\, \xi_2') \in S^{m-1}\times S^{n-1}$.

For any given $\Omega \in L\log^+ L(S^{m-1}\times S^{n-1})$ and $(\xi_1',\, \xi_2') \in S^{m-1}\times S^{n-1}$, set

$$E = \{(x_1',\, x_2') \in S^{m-1}\times S^{n-1} : |\Omega(x_1',\, x_2')| \le |\xi_1'\cdot x_1'|^{-1/2}|\xi_2'\cdot x_2'|^{-1/2}\}.$$

We write

$$\begin{aligned}&\int\!\!\int_{S^{m-1}\times S^{n-1}}|\Omega(x_1',\, x_2')|\left(\log\frac{1}{|\xi_1'\cdot x_1'|}\log\frac{1}{|\xi_2'\cdot x_2'|}\right)^{1/2} d\sigma(x_1')d\sigma(x_2') \\ = \;& \int\!\!\int_E |\Omega(x_1',\, x_2')|\left(\log\frac{1}{|\xi_1'\cdot x_1'|}\log\frac{1}{|\xi_2'\cdot x_2'|}\right)^{1/2} d\sigma(x_1')d\sigma(x_2') \\ &+\int\!\!\int_{E^c} |\Omega(x_1',\, x_2')|\left(\log\frac{1}{|\xi_1'\cdot x_1'|}\log\frac{1}{|\xi_2'\cdot x_2'|}\right)^{1/2} d\sigma(x_1')d\sigma(x_2') \\ := \;& I_1 + I_2.\end{aligned}$$

At first, we estimate I_1.

$$\begin{aligned}
I_1 &= \int\int_E |\Omega(x_1', x_2')| \left(\log\frac{1}{|\xi_1'\cdot x_1'|}\log\frac{1}{|\xi_2'\cdot x_2'|}\right)^{1/2} d\sigma(x_1')d\sigma(x_2') \\
&\le \int\int_{S^{m-1}\times S^{n-1}} \frac{1}{|\xi_1'\cdot x_1'|^{1/2}}\frac{1}{|\xi_2'\cdot x_2'|^{1/2}} \\
&\quad \times \left(\log\tfrac{1}{|\xi_1'\cdot x_1'|}\log\tfrac{1}{|\xi_2'\cdot x_2'|}\right)^{1/2} d\sigma(x_1')d\sigma(x_2') \\
&= 4\omega_{m-2}\omega_{n-2}\left[\int_0^{\pi/2}\frac{1}{|\cos\theta_1|^{1/2}}\left(\log\frac{1}{|\cos\theta_1|}\right)^{1/2}\sin^{m-2}\theta_1 d\theta_1\right] \\
&\quad \times \left[\int_0^{\pi/2}\frac{1}{|\cos\theta_2|^{1/2}}\left(\log\frac{1}{|\cos\theta_2|}\right)^{1/2}\sin^{n-2}\theta_2 d\theta_2\right] < \infty,
\end{aligned}$$

where θ_i denotes the angle of ξ_i' and x_i' $(i = 1, 2)$, ω_{N-2} denotes the Lebesgue measure of S^{N-1} $(N = m$ or $n)$.

Next we estimate I_2. Noting

$$|\Omega(x_1', x_2')| > \max\left\{\frac{1}{|\xi_1'\cdot x_1'|^{1/2}}, \quad \frac{1}{|\xi_2'\cdot x_2'|^{1/2}}\right\} \quad \text{for} \quad (x_1', x_2') \in E^c,$$

we have

$$\begin{aligned}
I_2 &\le C\int\int_{E^c} |\Omega(x_1', x_2')| \\
&\quad \times (\log|\Omega(x_1', x_2')|\log|\Omega(x_1', x_2')|)^{1/2} d\sigma(x_1')d\sigma(x_2') \\
&\le C\int\int_{S^{m-1}\times S^{n-1}} |\Omega(x_1', x_2')|\log^+|\Omega(x_1', x_2')|d\sigma(x_1')d\sigma(x_2') < \infty.
\end{aligned}$$

This proves Proposition 1. □

Proof of Theorem 2. By Plancherel's theorem, we have

$$\begin{aligned}
&\|\mu_\Omega(f)\|_2^2 \\
&= \int\int_{\mathbb{R}^m\times\mathbb{R}^n}\int_0^\infty\int_0^\infty |F_{s,t}(x_1, x_2)|^2\frac{dsdt}{s^3t^3}dx_1dx_2 \\
&= \int_0^\infty\int_0^\infty\int\int_{\mathbb{R}^m\times\mathbb{R}^n}\left|\widehat{\frac{1}{st}F_{s,t}}(\xi_1, \xi_2)\right|^2 d\xi_1 d\xi_2\frac{dsdt}{st} \\
&= \int_0^\infty\int_0^\infty\int\int_{\mathbb{R}^m\times\mathbb{R}^n}\left|\widehat{\sigma_{s,t} * f}(\xi_1, \xi_2)\right|^2 d\xi_1 d\xi_2\frac{dsdt}{st} \\
&= \int\int_{\mathbb{R}^m\times\mathbb{R}^n}\left[\int_0^\infty\int_0^\infty |\widehat{\sigma_{s,t}}(\xi_1, \xi_2)|^2\frac{dsdt}{st}\right]|\widehat{f}(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2,
\end{aligned}$$

where

$$\widehat{\sigma_{s,t}}(\xi_1,\, \xi_2) = \frac{1}{st}\int\int_{|y_1|\le s,\, |y_2|\le t} \frac{\Omega(y_1',\, y_2')}{|y_1|^{m-1}|y_2|^{n-1}} e^{-2\pi i(\xi_1\cdot y_1+\xi_2\cdot y_2)} dy_1 dy_2.$$

Thus, to prove Theorem 2, it suffices to show that

$$\int_0^\infty \int_0^\infty |\widehat{\sigma_{s,\,t}}(\xi_1,\, \xi_2)|^2 \frac{dsdt}{st} < \infty$$

holds uniformly for all $(\xi_1,\, \xi_2) \in \mathbb{R}^m \times \mathbb{R}^n$.

Note that $\widehat{\sigma_{s,\,t}}(\xi_1,\, \xi_2) = \widehat{\sigma_{1,\,1}}(s\xi_1,\, t\xi_2)$, we write

$$\begin{aligned}
&\int_0^\infty \int_0^\infty |\widehat{\sigma_{s,\,t}}(\xi_1,\, \xi_2)|^2 \frac{dsdt}{st} \\
= &\int_0^\infty \int_0^\infty |\widehat{\sigma_{1,\,1}}(s\xi_1,\, t\xi_2)|^2 \frac{dsdt}{st} = \int_0^\infty \int_0^\infty |\widehat{\sigma_{1,\,1}}(s\xi_1',\, t\xi_2')|^2 \frac{dsdt}{st} \\
= &\left[\int_0^1 \int_0^1 + \int_1^\infty \int_0^1 + \int_0^1 \int_1^\infty + \int_1^\infty \int_1^\infty\right] |\widehat{\sigma_{1,\,1}}(s\xi_1',\, t\xi_2')|^2 \tfrac{dsdt}{st} \\
:= &J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

For J_1, by the vanishing property (1.1), we have

$$\begin{aligned}
J_1 &= \int_0^1 \int_0^1 \Bigg| \int\int_{|y_1|\le 1,\, |y_2|\le 1} \frac{\Omega(y_1',\, y_2')}{|y_1|^{m-1}|y_2|^{n-1}} \\
&\quad \times e^{-2\pi i(s\xi_1'\cdot y_1 + t\xi_2'\cdot y_2)} dy_1 dy_2 \Bigg| \tfrac{dsdt}{st} \\
&= \int_0^1 \int_0^1 \Bigg| \int_0^1 \int_0^1 \int\int_{S^{m-1}\times S^{n-1}} \Omega(y_1',\, y_2') \left[e^{-2\pi i r_1 s\xi_1'\cdot y_1'} - 1\right] \\
&\quad \times \left[e^{-2\pi i r_2 t\xi_2'\cdot y_2'} - 1\right] d\sigma(y_1') d\sigma(y_2') dr_1 dr_2 \Bigg|^2 \tfrac{dsdt}{st} \\
&\le C\|\Omega\|^2_{L^1(S^{m-1}\times S^{n-1})},
\end{aligned}$$

where C is independent of $(\xi_1,\, \xi_2) \in \mathbb{R}^m \times \mathbb{R}^n$.

To estimate J_2, for $s \in [0,\, 1]$ and $\xi_1' \in S^{m-1}$, we denote

$$E_1 = \{y_1' \in S^{m-1} :\ \ |\xi_1' \cdot y_1'|^{-1} \ge s^{1/2}\} \times S^{n-1}$$

and

$$E_2 = \{y_1' \in S^{m-1} :\ \ |\xi_1' \cdot y_1'|^{-1} < s^{1/2}\} \times S^{n-1}.$$

Then

$$\begin{aligned}
J_2 &= \int_1^\infty \int_0^1 \bigg| \int\int_{|y_1|\le 1,\, |y_2|\le 1} \frac{\Omega(y_1',\, y_2')}{|y_1|^{m-1}|y_2|^{n-1}} \\
&\quad \times e^{-2\pi i(s\xi_1'\cdot y_1 + t\xi_2'\cdot y_2)} dy_1 dy_2 \bigg| \frac{dsdt}{st} \\
&= \int_1^\infty \int_0^1 \bigg| \int\int_{S^{m-1}\times S^{n-1}} \Omega(y_1',\, y_2') \bigg(\int_0^1 \int_0^1 e^{-2\pi i s r_1 \xi_1'\cdot y_1'} \\
&\quad \times \Big[e^{-2\pi i r_2 t \xi_2'\cdot y_2'} - 1 \Big] dr_1 dr_2 \bigg) d\sigma(y_1') d\sigma(y_2') \bigg|^2 \frac{dsdt}{st} \\
&\le 2\int_1^\infty \int_0^1 \bigg| \int\int_{E_1} \Omega(y_1',\, y_2') \bigg(\int_0^1 \int_0^1 e^{-2\pi i s r_1 \xi_1'\cdot y_1'} \\
&\qquad\qquad \times \Big[e^{-2\pi i r_2 t \xi_2'\cdot y_2'} - 1 \Big] dr_1 dr_2 \bigg) d\sigma(y_1') d\sigma(y_2') \bigg|^2 \frac{dsdt}{st} \\
&\quad + 2\int_1^\infty \int_0^1 \bigg| \int\int_{E_2} \Omega(y_1',\, y_2') \bigg(\int_0^1 \int_0^1 e^{-2\pi i s r_1 \xi_1'\cdot y_1'} \\
&\qquad\qquad \times \Big[e^{-2\pi i r_2 t \xi_2'\cdot y_2'} - 1 \Big] dr_1 dr_2 \bigg) d\sigma(y_1') d\sigma(y_2') \bigg|^2 \frac{dsdt}{st} \\
&:= 2(J_{21} + J_{22}).
\end{aligned}$$

Let

$$A(\xi_1', \xi_2',\, y_1', y_2',\, s, t) = \int_0^1 \int_0^1 e^{-2\pi i s r_1 \xi_1'\cdot y_1'} \Big[e^{-2\pi i r_2 t \xi_2'\cdot y_2'} - 1 \Big] dr_1 dr_2.$$

Then, for J_{21}, we have

$$\begin{aligned}
J_{21} &= \int_1^\infty \int_0^1 \bigg| \int\int_{E_1} \Omega(y_1',\, y_2') \\
&\quad \times A(\xi_1', \xi_2',\, y_1', y_2',\, s, t) d\sigma(y_1') d\sigma(y_2') \bigg|^2 \frac{dsdt}{st} \\
&= \int_1^\infty \int_0^1 \bigg| \int\int_{S^{m-1}\times S^{n-1}} \Omega(y_1',\, y_2') \chi_{E_1}(y_1',\, y_2') \\
&\quad \times A(\xi_1', \xi_2',\, y_1', y_2',\, s, t) d\sigma(y_1') d\sigma(y_2') \bigg|^2 \frac{dsdt}{st} \\
&\le \bigg\{ \int\int_{S^{m-1}\times S^{n-1}} |\Omega(y_1',\, y_2')| \bigg(\int_1^\infty \int_0^1 \chi_{E_1}(y_1',\, y_2')| \\
&\quad \times A(\xi_1', \xi_2',\, y_1', y_2',\, s, t)|^2 \frac{dsdt}{st} \bigg)^{1/2} d\sigma(y_1') d\sigma(y_2') \bigg\}^2 \\
&\le \bigg\{ \int\int_{S^{m-1}\times S^{n-1}} |\Omega(y_1', y_2')| \bigg(\int_1^{|\xi_1'\cdot y_1'|^{-2}} \int_0^1 \bigg| \int_0^1 e^{-2\pi i s r_1 \xi_1'\cdot y_1'} dr_1' \bigg|^2 \\
&\quad \times \bigg| \int_0^1 \Big[e^{-2\pi i t r_2 \xi_2'\cdot y_2'} - 1 \Big] dr_2 \bigg|^2 \frac{dsdt}{st} \bigg)^{1/2} d\sigma(y_1') d\sigma(y_2') \bigg\}^2
\end{aligned}$$

$$\leq C\left\{\int\int_{S^{m-1}\times S^{n-1}}|\Omega(y_1',\,y_2')|\left(\log\frac{1}{|\xi_1'\cdot y_1'|}\right)^{1/2}d\sigma(y_1')d\sigma(y_2')\right\}^2$$
$$\leq C.$$

For J_{22}, note that

$$\left|\int_0^1 e^{-2\pi i s r_1\xi_1'\cdot y_1'}dr_1\right|\leq\left(\frac{1}{s|\xi_1'\cdot y_1'|}\right)^{1/2}$$

and

$$\left|\int_0^1\left[e^{-2\pi i t r_2\xi_2'\cdot y_2'}-1\right]dr_2\right|\leq Ct,$$

we have

$$\begin{aligned}J_{22}=&\left\{\int\int_{S^{m-1}\times S^{n-1}}|\Omega(y_1',\,y_2')|\right.\\&\times\left(\int_1^\infty\int_0^1\chi_{E_2}(y_1',y_2')|A(\xi_1',\xi_2',\,y_1',y_2',\,s,t)|^2\right.\\&\left.\left.\times\tfrac{dsdt}{st}\right)^{1/2}d\sigma(y_1')d\sigma(y_2')\right\}^2\\&\leq C\left\{\int\int_{S^{m-1}\times S^{n-1}}|\Omega(y_1',\,y_2')|\left[\left(\int_1^\infty\chi_{E_2}(y_1',\,y_2')\frac{1}{s^2|\xi_1'\cdot y_1'|}ds\right)\right.\right.\\&\left.\left.\times\left(\int_0^1 t^2\frac{dt}{t}\right)\right]^{1/2}d\sigma(y_1')d\sigma(y_2')\right\}^2\\&\leq C\left\{\int\int_{S^{m-1}\times S^{n-1}}|\Omega(y_1',\,y_2')|\left(\int_1^\infty\frac{1}{s^{1+1/2}}ds\right)^{1/2}d\sigma(y_1')\sigma(y_2')\right\}^2\\&\leq C\|\Omega\|^2_{L^1(S^{m-1}\times S^{n-1})}\leq C.\end{aligned}$$

Thus $J_2\leq C$.

Similarly, we can conclude that $J_3\leq C$.

It remains to estimate J_4. For $s,\,t\in[1,\,\infty)$ and $(\xi_1',\,\xi_2')\in S^{m-1}\times S^{n-1}$, set

$$D_1=\{y_1'\in S^{m-1}:\ \ |\xi_1'\cdot y_1'|^{-1}\geq s^{1/2}\}\times\{y_2'\in S^{n-1}:\ \ |\xi_2'\cdot y_2'|^{-1}\geq t^{1/2}\},$$

$$D_2=\{y_1'\in S^{m-1}:\ \ |\xi_1'\cdot y_1'|^{-1}\geq s^{1/2}\}\times\{y_2'\in S^{n-1}:\ \ |\xi_2'\cdot y_2'|^{-1}< t^{1/2}\},$$

$$D_3=\{y_1'\in S^{m-1}:\ \ |\xi_1'\cdot y_1'|^{-1}< s^{1/2}\}\times\{y_2'\in S^{n-1}:\ \ |\xi_2'\cdot y_2'|^{-1}\geq t^{1/2}\},$$

$$D_4=\{y_1'\in S^{m-1}:\ \ |\xi_1'\cdot y_1'|^{-1}< s^{1/2}\}\times\{y_2'\in S^{n-1}:\ \ |\xi_2'\cdot y_2'|^{-1}< t^{1/2}\}.$$

Then

$$\begin{aligned} J_4 \le 4\Bigg\{&\int_1^\infty\int_1^\infty\left|\int\int_{D_1}\Omega(y_1',\,y_2')B(\xi_1',\xi_2,'\,y_1',y_2',\,s,t)d\sigma(y_1')d\sigma(y_2')\right|^2\frac{dsdt}{st}\\ &+\int_1^\infty\int_1^\infty\left|\int\int_{D_2}\Omega(y_1',\,y_2')B(\xi_1',\xi_2,'\,y_1',y_2',\,s,t)d\sigma(y_1')d\sigma(y_2')\right|^2\frac{dsdt}{st}\\ &+\int_1^\infty\int_1^\infty\left|\int\int_{D_3}\Omega(y_1',\,y_2')B(\xi_1',\xi_2,'\,y_1',y_2',\,s,t)d\sigma(y_1')d\sigma(y_2')\right|^2\frac{dsdt}{st}\\ &+\int_1^\infty\int_1^\infty\left|\int\int_{D_4}\Omega(y_1',\,y_2')B(\xi_1',\xi_2,'\,y_1',y_2',\,s,t)d\sigma(y_1')d\sigma(y_2')\right|^2\frac{dsdt}{st}\Bigg\}\\ :=&\, 4(J_{41}+J_{42}+J_{43}+J_{44}), \end{aligned}$$

where

$$B(\xi_1',\xi_2',\,y_1',y_2',\,s,t)=\int_0^1\int_0^1 e^{-2\pi[sr_1\xi_1'\cdot y_1'+tr_2\xi_2'\cdot y_2']}dr_1dr_2.$$

Similarly to estimating J_{21} and J_{22}, we easily obtain that

$$\begin{aligned} J_{41}&\le C\left\{\int\int_{S^{m-1}\times S^{n-1}}|\Omega(y_1',\,y_2')|\left(\log\frac{1}{|\xi_1'\cdot y_1'|}\log\frac{1}{|\xi_2'\cdot y_2'|}\right)^{1/2}d\sigma(y_1')d\sigma(y_2')\right\}^2\\ &\le C; \end{aligned}$$

$$\begin{aligned} J_{42}&\le C\left\{\int\int_{S^{m-1}\times S^{n-1}}|\Omega(y_1',\,y_2')|\left(\log\frac{1}{|\xi_1'\cdot y_1'|}\int_1^\infty\frac{1}{t^{3/2}}dt\right)^{1/2}d\sigma(y_1')d\sigma(y_2')\right\}^2\\ &\le C; \end{aligned}$$

$$\begin{aligned} J_{43}&\le C\left\{\int\int_{S^{m-1}\times S^{n-1}}|\Omega(y_1',\,y_2')|\left(\log\frac{1}{|\xi_2'\cdot y_2'|}\int_1^\infty\frac{1}{s^{3/2}}ds\right)^{1/2}d\sigma(y_1')d\sigma(y_2')\right\}^2\\ &\le C; \end{aligned}$$

$$\begin{aligned} J_{44}&\le C\left\{\int\int_{S^{m-1}\times S^{n-1}}|\Omega(y_1',\,y_2')|\left(\int_1^\infty\frac{1}{s^{3/2}}ds\int_1^\infty\frac{1}{t^{3/2}}dt\right)^{1/2}d\sigma(y_1')d\sigma(y_2')\right\}^2\\ &\le C. \end{aligned}$$

This completes the proof of Theorem 2. □

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